Universal and Tight Bounds on Counting Errors of Count-Min Sketch with Conservative Updates

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Outline

1) Streaming Algorithms

2) Classical Streaming Algorithms

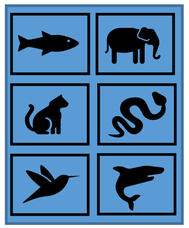
3) Count-Min Sketch with Conservative Updates

4) Research Gap and Motivation

5) Contribution

Streaming Algorithms

Catalog of items $\mathcal{I} = \{1, \ldots, N\}$



Data stream $r_1, \ldots, r_T$

Objective: Answer queries

1. The count of item i?
2. Top k items?
3. $\phi -$ heavy hitters?
4. Distinct items?

$$n_i(t) = \sum_{s=1}^{t-1} I(r_s = i)$$

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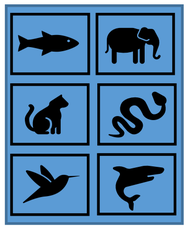

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Constraints

1. High arrival rate, e.g., 1M item/s
2. Limited memory

$\hat{\mathbf{n}}(t)$ using $o(N)$ memory

And one pass algorithm

Network monitoring + clickstream analytics

Classical Streaming Algorithms

Counter-based

Id + count for a subset of items, e.g., Space Saving [4], Frequent [1,2]

[1]Misra, S., & Gries, D. (1982). *Finding repeated elements*. **Science of Computer Programming**, 2(2), 143–152.
[2] Demaine, E. D., López-Ortiz, A., & Munro, J. I. (2002). *Frequency estimation of internet packet streams with limited space*. In **European Symposium on Algorithms (ESA)** (pp. 348–360).
[3] Charikar, M., Chen, K., & Farach-Colton, M. (2002). *Finding frequent items in data streams*. In **International Colloquium on Automata, Languages and Programming (ICALP)** (pp. 693–703).
[4] Metwally, A., Agrawal, D., & El Abbadi, A. (2005). *Efficient computation of frequent and top-k elements in data streams*. In **International Conference on Database Theory (ICDT)** (pp. 398–412).
[5] Cormode, G., & Muthukrishnan, S. (2005). *An improved data stream summary: The Count–Min sketch and its applications*. **Journal of Algorithms**, 55(1), 58–75.

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Sketch-based

Random projection of counts vector, e.g., Count-Sketch [3], Count-Min-Sketch [5].

$$(Y_1(t), \ldots, Y_M(t)) = \mathbf{n}(t)\mathbf{\Pi} : \ \mathbf{\Pi} \in \mathcal{M}_{N,M}(\mathbb{R})$$

Sparse random matrix

[1]Misra, S., & Gries, D. (1982). *Finding repeated elements*. **Science of Computer Programming**, 2(2), 143–152.
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Count-Min Sketch (CMS)

$$\mathbf{Y}(t) = \mathbf{n}(t) \cdot \mathbf{\Pi}$$

$$\mathbf{\Pi} = (\pi_{i,j}) \in \mathcal{M}_{N,M}(\{0,1\}) : \sum_{j=1}^{M} \pi_{i,j} = d \ll M$$

Popularity estimation

$$\hat{n}_i(t) = \min_{j \in [M]:\, \pi_{i,j}=1} Y_j(t)$$

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Example:

$N = 5,\ M = 4,\ d = 2$

$$\mathbf{n} = [10 \quad 10 \quad 1 \quad 1 \quad 1] \qquad \mathbf{\Pi} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$\sum_{i \in \mathcal{N}(4)} n_i = n_2 + n_4$$

$$\mathbf{Y} = [21 \quad 2 \quad 12 \quad 11]$$

$$\hat{n}_1 = \min(21, 12) = 12$$

Count-Min Sketch (CMS)

Implicit storage via d hash functions: $\forall l \in [d],\ h_l : \mathcal{I} \mapsto [M]$

$$\pi_{i,j} = 1 \iff \exists l \in [d] :\ h_l(i) = j$$

Example

$$\mathbf{Y}(t) = [21 \quad 2 \quad 12 \quad 11] \longrightarrow \mathbf{Y}(t+1) = [22 \quad 2 \quad 12 \quad 12]$$

$$h_1(r_{t+1}) = 1 \qquad h_2(r_{t+1}) = 4$$

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Update Rule

$$Y_c(t+1) = \begin{cases} Y_c(t+1) + 1, & \text{if } \exists l \in [d] :\ h_l(r_{t+1}) = c, \\ Y_c(t), & \text{otherwise.} \end{cases}$$

Count Estimation

$$\hat{n}_i(t) = \min_{l \in [d]} Y_{h_l(i)}(t)$$

CMS with Conservative Updates (CMS-CU)

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$$Y_c(t+1) = \begin{cases} Y_c(t+1) + 1, & \text{if } c \in \operatorname{argmin}_{l \in [d]} Y_{h_l(r_{t+1})}(t+1), \\ Y_c(t), & \text{otherwise.} \end{cases}$$

Count Estimation

$$\hat{n}_i(t) = \min_{l \in [d]} Y_{h_l(i)}(t)$$

Counting Error (CE)

$$e_i(t) = \hat{n}_i(t) - n_i(t)$$

Research Gap and Motivation

Quantifying CE of infrequent Items is crucial: Risk of misclassification as Heavy Hitters

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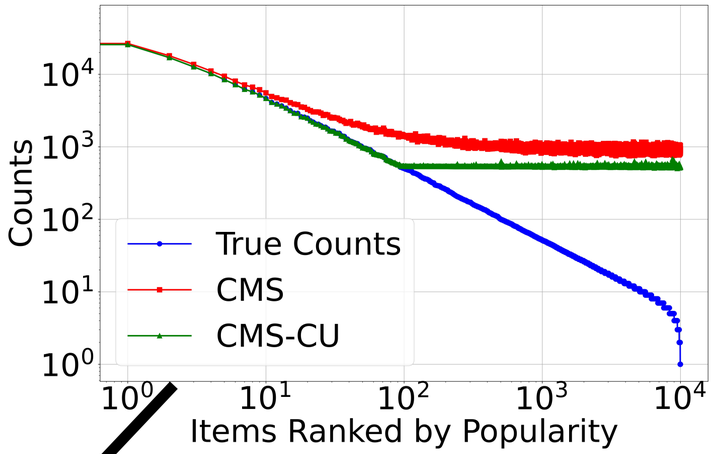
Stream: $N = 10^4, T = 5 * 10^5, p_i \sim 1/i$

Sketch: $(M, d) = (1000,5)$



Inflated Count ≈ CE of infrequent items

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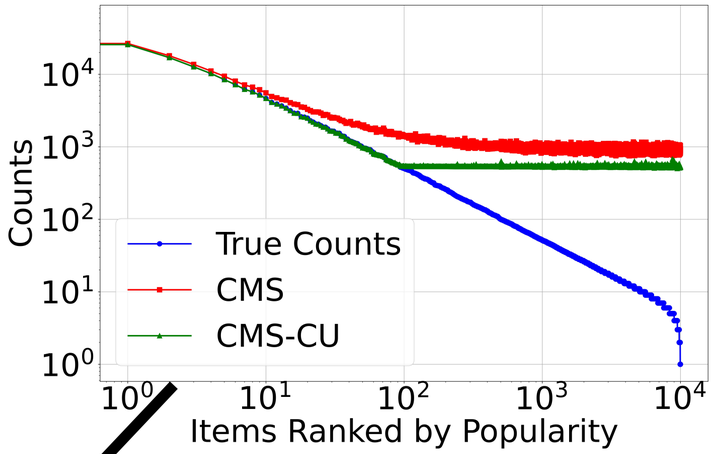

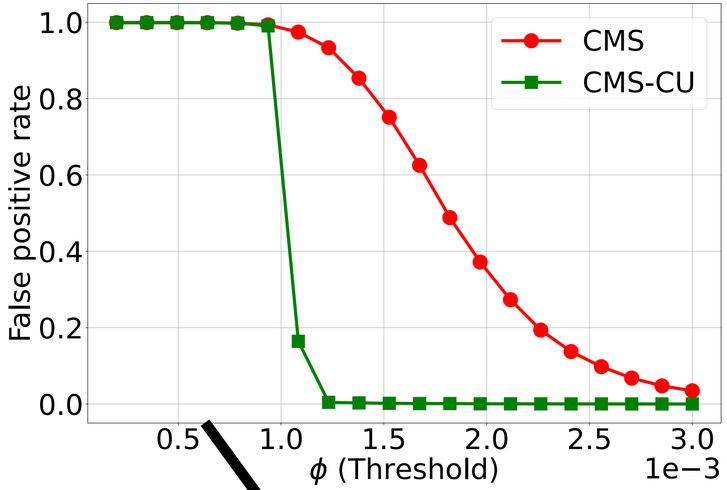

Inflated Count $\approx$ CE of infrequent items $\sim$ Critical Threshold ϕ_c

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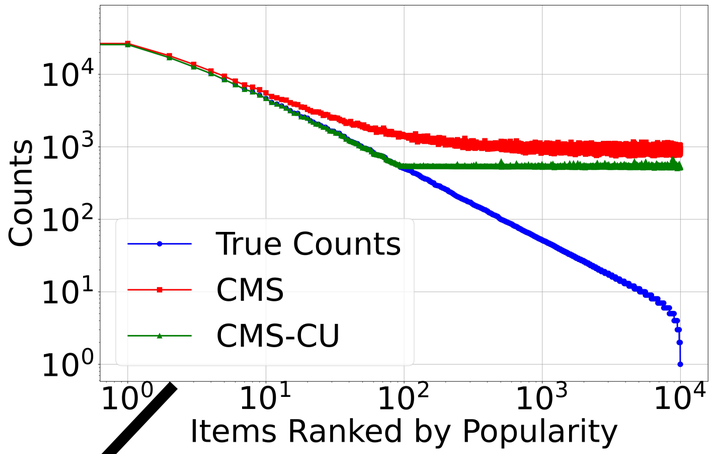

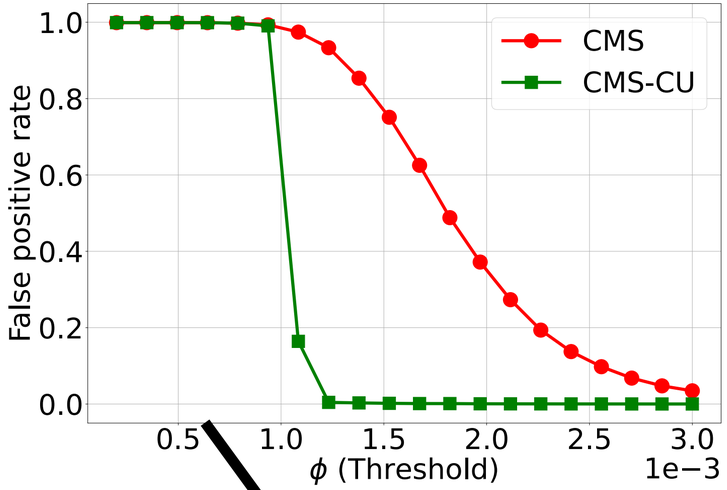

Inflated Count ≈ CE of infrequent items $\sim$ Critical Threshold ϕ_c

Quantifying CE of infrequent items → Near-optimal Configuration of CMS-CU

Contribution: Definitions

Definition 1 [Counting Error (CE)]

$$e_i(t) \triangleq \hat{n}_i(t) - n_i(t)\,.$$

Definition 2 [Unseen Items' Error (UE)]

$$e^0(t) \triangleq \min_{c \in \mathbf{X}} Y_c(t) : \ \mathbf{X} \sim \text{Unif}(\binom{[M]}{d})\,.$$

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Definition 3 [Reference Error (RE)]

$$e^*(t) \triangleq \min_{c \in \mathbf{X}} \tilde{Y}_c(t),$$

$$\tilde{Y}_c(t+1) = \begin{cases} \tilde{Y}_c(t) + 1, & \text{if } c \in \text{argmin}_{u \in \mathbf{X}_{t+1}} \tilde{Y}_u(t), \\ \tilde{Y}_c(t), & \text{otherwise.} \end{cases}$$

, (X_t) and $\mathbf{X}$ are i.i.d.

Contribution: RE is important

Theorem 1 [From CE to UE]:

$$\Pr\left(e_i(t) \geq x\right) \leq \Pr\left(e^0(t) \geq x\right)$$

1) UE is a proxy for the errors of infrequent items

2) The false positive rate in ϕ-Heavy Hitters can be bounded in terms of UE

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Theorem 2 [From UE to RE]: For a class of skewed streams: $\mathrm{E}\left[e^0(t)\right] \leq C_k + \alpha_k \mathrm{E}\left[\hat{e}^{*,k}(t)\right]$

	Counters	Stream
$e^*(T)$	$[M]$	$\mathbf{r} = (r_t)_{t \in [T]}$
$\hat{e}^{*,k}(T)$	$[M] \setminus \cup_{i=1}^{k} \mathbf{h}(i)$	$\mathbf{r} \setminus$ Top k

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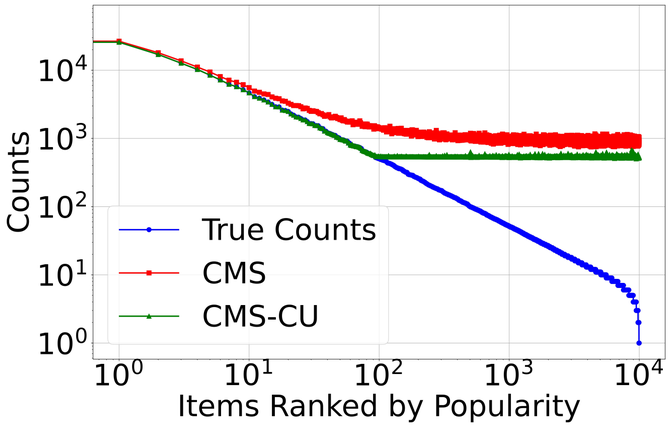

1) These bounds are provably tight for some streams

2) These bounds are valid for any stream if:

Conjecture 1: $\mathrm{E}\left[\hat{e}^{0,k}(t)\right] \leq \mathrm{E}\left[\hat{e}^{*,k}(t)\right]$

Computing RE is important

Contribution: Computing RE

Analyzing $\tilde{Y}$ is difficult

$$\tilde{Y}_c(t+1) = \begin{cases} \tilde{Y}_c(t) + 1, & \text{if } c \in \text{argmin}_{u \in \mathbf{X}_{t+1}} \tilde{Y}_u(t), \\ \tilde{Y}_c(t), & \text{otherwise.} \end{cases}$$

1) Complex dependency on $\mathbf{X}_{t+1}$, vs CMS: $Y_c(t) = \sum_{s=1}^{t} I\left(c \in \mathbf{X}_s\right)$

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Key Ideas

1) Alternate representation:

$$\mathbf{\Delta}(t): \ \Delta_l(t) \triangleq \sum_{c=1}^{m} I\left(\tilde{Y}_c(t) - \tilde{Y}_{\min}(t) = l\right).$$

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2) Variants of $\tilde{Y}$:

CU-LB ($\tilde{\mathbf{Y}}^{\text{lb},g}$) and CU-UB ($\tilde{\mathbf{Y}}^{\text{ub},g}$)

2.1) $(\mathbf{\Delta}^g(t))_t$ is a finite state space Markov chain of size $\binom{m+g-d}{g}$

2.2) $\tilde{\mathbf{Y}}^{\text{lb},g} \leq \tilde{\mathbf{Y}} \leq \tilde{\mathbf{Y}}^{\text{ub},g}$

Contribution: Computing RE

If $\mathbf{X}_{t+1} \subset \text{argmax}_{c \in [m]} Y_c^{\text{lb,g}}(t)$, and $G^{\text{lb},g}(t) = g$

$\longrightarrow$ $\tilde{\mathbf{Y}}^{\text{lb},g}(t+1) = \tilde{\mathbf{Y}}^{\text{lb},g}(t)$

If $\mathbf{X}_{t+1} \subset \text{argmax}_{c \in [m]} Y_c^{\text{ub,g}}(t)$, and $G^{\text{ub},g}(t) = g$

$\longrightarrow$ Increment all min value counters and those in $\mathbf{X}_{t+1}$

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Otherwise:

$\rightarrow$ $\tilde{\mathbf{Y}}^{\text{lb},g}(t+1) = \text{CU}\left(\tilde{\mathbf{Y}}^{\text{lb},g}(t)\right)$

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Theorem 3:

1) $\ell_g < \ell_{g+1} \leq \mathrm{E}\left[e^*(T)/T\right] \leq U_{g+1} < U_g$

2) ℓ_g and U_g can be computed in $\mathcal{O}\left((T+g)m\binom{m+g-d}{g}\right)$

3) $\lim_{t\to\infty} \Pr\left(\ell_g(\infty) \leq e^*(t)/t \leq U_g(\infty)\right) = 1$

Table 1: Results for different values of g with $m = 50$, $d = 4$, and $T = 250$.

g	1	2	3	4	5
$\ell_g(T)$	0.01860	0.02956	0.03420	0.03540	0.03559
$U_g(T)$	0.07654	0.04090	0.03637	0.03572	0.03562
Execution time in (s)	0.05	0.68	17	257	3241

Thank you.

CMS-CU: Previous studies

[6] Uniform Counters selection via mean-field analysis: Solving a large system of ODES

[7,8] Assumes stationary Stream: Analysis captures errors for popular items only

[9,10] Uniform and phase transition

[6] Bianchi, G., Duffy, K., Leith, D., & Shneer, V. (**2012**). Modeling conservative updates in multi-hash approximate count sketches. In *24th International Teletraffic Congress (**ITC**)* (pp. 1–8).

[7] Einziger, G., & Friedman, R. (**2015**). A formal analysis of conservative update based approximate counting. In *2015 International Conference on Computing, Networking and Communications (**ICNC**)* (pp. 255–259)

[8] Ben Mazziane, Y., Alouf, S., & Neglia, G. (**2022**). Analyzing count–min sketch with conservative updates. ***Computer Networks***, 217, 109315.

[9] Fusy, É., & Kucherov, G. (**2023**). Phase transition in count approximation by count–min sketch with conservative updates. In *International Conference on Algorithms and Complexity (**CIAC**)* (pp. 232–246).

[10] Fusy, É., & Kucherov, G. (**2023**). Count–min sketch with variable number of hash functions: an experimental study. In *String Processing and Information Retrieval (**SPIRE**)* (pp. 218–232).